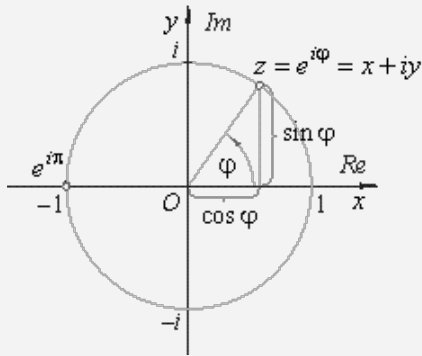


Complex Unit Circle



DFT Table

Table C.2 Fourier Transform Pairs	Table C.3 Fourier Transform Theorems																																																						
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Discrete Fourier Transform

$$X_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i k n / N}, \quad k \in \mathbb{Z}$$

Butterworth Lowpass Filter

$$H(u, v) = \frac{1}{1 + \left(\frac{D(u, v)}{D_0} \right)^{2n}}$$

D_0 is cutoff freq and $D(u, v)$ is distribution of (u, v) from centered origin. n is order

Sinc Definition

$$\text{sinc}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases}$$

Wrap Around Error

Solved by zero padding

If $f(x)$ and $h(x)$ are A and B samples respectively, pad $f(x)$ and $h(x)$ with zeros so both have length $P \geq A+B-1$

If not zero, creates discontinuity called "frequency leakage", equivalent to convolving with $\text{sinc}()$ function

Reduced by multiplying with function that tapers smoothly to zero (windowing or apodizing)

Steps for Filtering

- 1 + 2. Given $f(x, y)$ is $M \times N$, zero pad to $2M \times 2N$ ($P \times Q$)
3. Multiply by $(-1)^{x+y}$ to center
4. Take DFT of $f(x, y)$ to get $F(u, v)$
5. Generate symmetric filter $H(u, v)$ of size $P \times Q$
6. Get processed image $g(x, y) = \{ \text{real} [F^{-1} \{ G(u, v) \}] * (-1)^{x+y} \}$

Fourier Spectrum and Phase Angle

$$\begin{aligned} P(u, v) &= |F(u, v)|^2 \\ |F(u, v)| &= \sqrt{|F(u, v)|^2} = \sqrt{|F(u, v)|^2} \\ \phi(u, v) &= \tan^{-1} \left[\frac{F_{\text{imag}}(u, v)}{F_{\text{real}}(u, v)} \right] \end{aligned}$$

Conjugate Symetry

- $F^*(u, v) = F(-u, -v)$ (Conjugate Symmetry)
- $F^*(-u, -v) = -F(u, v)$ (Conjugate Asymmetry)

Spatial Shift Theorem

$$\mathcal{F} \{ f(t - t_0) \} (s) = e^{-j2\pi s t_0} F(s)$$

Spatial transform only affects FT phase

2D Convolution

$$g(m, n) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f(k, l) h(m-k, n-l)$$

Convolution Theorem

$$\begin{aligned} \mathcal{F}(f * g) &= \mathcal{F}(f) \cdot \mathcal{F}(g) \\ \mathcal{F}(f \cdot g) &= \mathcal{F}(f) * \mathcal{F}(g) \\ f * g &= \mathcal{F}^{-1} \{ \mathcal{F}(f) \cdot \mathcal{F}(g) \} \\ f \cdot g &= \mathcal{F}^{-1} \{ \mathcal{F}(f) * \mathcal{F}(g) \} \end{aligned}$$

Space convolution = frequency multiplication

Center DC

To shift $F(0, 0)$ (DC Component) to center, multiply by $(-1)^{x+y}$

Frequency Shift Theorem

$$\mathcal{F} \{ e^{-j2\pi f_0 t} f(t) \} = F(f - f_0)$$

Power Spectrum

$$P(u, v) = |F(u, v)|^2$$

Total power of image is just sum of $P(u, v)$ over $P-1, Q-1$

$$a = 100 [\text{double sum } P(u, v) / P_t]$$

Convolution Definition

$$(f * g)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} f[m] g[n-m]$$

Unsharp, Highboost, High-Emphasis

$$g(x, y) = \frac{1}{k} \{ 1 + k H_{HP}(u, v) \} F(u, v)$$

$g_{\text{mask}}(x, y) = f(x, y) - \text{flp}(x, y)$
 $g(x, y) = f(x, y) + k * g_{\text{mask}}(x, y)$
 $k=1$, unsharp
 $k>1$, highboost

Impulse Train Definition

$$\psi_T(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

2D DFT Definition

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi \left(\frac{k}{M} m + \frac{l}{N} n \right)}$$

2D Continuous Fourier Transform

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx dy$$

Fourier Series Definition

$$F_k(x) = a_k + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)).$$

Laplacian in Freq. Domain

$$\begin{aligned} H(u,v) &= -4\pi^2(u^2 + v^2) \\ H(u,v) &= -4\pi^2 \left[\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2 \right] \\ H(u,v) &= -4\pi^2 D^2 + (u,v) \\ &\rightarrow \nabla^2 f(x,y) = \mathcal{F}^{-1} \{ H(u,v) \mathcal{F} \{ f(x,y) \} \} \end{aligned}$$

Displacement: $g(x,y) = f(x,y) + \alpha \nabla^2 f(x,y)$, $\alpha = -1$

$$\begin{aligned} g(x,y) &= \mathcal{F}^{-1} \{ \mathcal{F} \{ f(x,y) \} H(u,v) \} \\ g(x,y) &= \mathcal{F}^{-1} \{ [1 - H(u,v)] \mathcal{F} \{ f(x,y) \} \} \\ g(x,y) &= \mathcal{F}^{-1} \{ [1 + 4\pi^2 D^2] \mathcal{F} \{ f(x,y) \} \} \end{aligned}$$

DC Component

$$F(0,0) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) = MN \bar{f}(x,y)$$

2D Sampling

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - nX) \delta(y - mY)$$

Gaussian Filter

$$\begin{aligned} \text{Low-Pass: } H(u) &= A e^{-\frac{u^2}{2\sigma^2}} \\ H(x) &= \sqrt{2\pi\sigma_x\sigma_y} e^{-\frac{\pi^2}{2\sigma_x^2}x^2 - \frac{\pi^2}{2\sigma_y^2}y^2} \\ \text{High-Pass (wide narrow band): } H(u) &= A e^{\frac{u^2}{2\sigma^2}} - B e^{\frac{u^2}{2\sigma^2}} \\ H(x) &= \sqrt{2\pi\sigma_x\sigma_y} e^{-\frac{\pi^2}{2\sigma_x^2}x^2 - \frac{\pi^2}{2\sigma_y^2}y^2} - \sqrt{2\pi\sigma_x\sigma_y} e^{-\frac{\pi^2}{2\sigma_x^2}x^2 - \frac{\pi^2}{2\sigma_y^2}y^2} \end{aligned}$$

C

By **Sawyer McLane**
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