

Algorithms			Running Time (cont)			In Place Sorting (cont)			QuickSort (cont)		
Definition	unambiguous procedure executed in a finite number of steps		Worst Case:	time on most difficult input	good for safety critical systems, easy to estimate	Importance:	Great for large data sets that take up large amounts of memory		> if (left < right) then exchange A[left] \$ A[right] exchange A[stop] \$ A[left] return left		
What makes a good algorithm?	Correctness, Speed, Space, Simplicity		Loop Invariants			Examples:	Selection Sort, Insertion Sort (Only moving elements around the array)		Time Complexities: • Worse case: – Already sorted array (either increasing or decreasing) – T(n) = T(n-1) + c n + d – T(n) is O(n2) • Average case: If the array is in random order, the pivot splits the array in roughly equal parts, so the average running time is O(n log n) • Advantage over mergeSort: – constant hidden in O(n log n) are smaller for quickSort. Thus it is faster by a constant factor – QuickSort is easy to do “in-place”		
Speed:	time it takes to solve problem		Definition			MergeSort:	Not in place: new array required				
Space:	amount of memory required		Steps:			QuickSort					
Simplicity:	easy to understand, easy to implement, easy to debug, modify, update		1. Initialization	If it is true prior to the iteration of the loop		Algorithm partition(A, start, stop) Input: An array A, indices start and stop. Output: Returns an index j and rearranges the elements of A such that for all i<j, A[i] ! A[j] and for all k>i, A[k] " A[j]. pivot # A[stop] left # start right # stop - 1 while left ! right do while left ! right and A[left] ! pivot) do left # left + 1 while (left ! right and A[right] " pivot) do right # right -1					
Running Time			2. Maintenance	If it is true before an iteration of the loop, it remains true before the next iteration							
Definition	measurement of the speed of an algorithm		3. Termination	When the loop terminates, the invariant gives us a useful property that helps show the algorithm is correct							
Dependent variables:	size of input & content of input										
Best Case:	time on the easiest input of fixed size	meaningless							QuickSort		
Average Case:	time on average input	good measure, hard to calculate							Divide:	choose an element of the array for pivot	
										divide into 3 sub-groups; those smaller, those larger and those equal to pivot	
									Conquer	recursively sort each group	
									Combine	concatenate the 3 lists	

Proofs by Induction (Examples)

Claim: for any $n \geq 1$, $1 + 2 + 3 + 4 + \dots + n = \frac{n \cdot (n+1)}{2}$

Proof:

• Base case: $n = 1$ $1 = \frac{1 \cdot 2}{2}$ ✓

• Induction step:

for any $k \geq 1$, if $1 + 2 + 3 + 4 + \dots + k = \frac{k \cdot (k+1)}{2}$

then $1 + 2 + 3 + 4 + \dots + k + (k+1) = \frac{(k+1) \cdot (k+2)}{2}$

Object Orientated Programming

Definition: User defined types to complement primitive types

Called a class

Contains: Data & methods

Static members: members shared by all objects of the class

Recursion Programming

Definition: using methods that call themselves

Structure:

base case: a simple occurrence that can be answered directly

recursive case: A more complex occurrence of the problem that cannot be directly answered, but can instead be described in terms of smaller occurrences of the same problem.

Divide & Conquer

Divide: the problem into sub problems that are similar to the original but smaller in size

Conquer: the sub-problems by solving them recursively. If they are small enough, solve them in a straightforward manner

Combine: the solutions to create a solution to the original problem

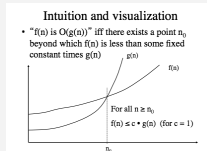
BIG O Definition

$f(n)$ & $g(n)$ are two non negative functions defined on the natural numbers N

$f(n)$ is $O(g(n))$ if there exists an integer n_0 and a real number c such that $\forall n \geq n_0$ only if: $f(n) \leq c \cdot g(n)$

N.B. The constant c must not depend on n

Big O Visualization



Big O Recurrence

Algorithm MergeSort(A, start, stop) Prim Op
if (start == stop) return 1 + 1 + 1 = 3
else
mid = (start + stop) / 2
MergeSort(A, start, mid) 1 + 1 + 1 = 3
MergeSort(A, mid + 1, stop) 1 + 1 + 1 = 3
Merge(A, start, mid, stop) 1 + 1 + 1 = 3
Let T(n) be # prim op by MergeSort when stop - start = n
T(n) = 3 + 2T(n/2) for n > 1
T(1) = 3
Goal: find an explicit formula of T
For illustrative purposes, suppose T(n) = 3 + 2T(n/2)
T(n) = 3 + 2(3 + 2T(n/2)) = 3 + 2n + 4T(n/2)
= 3 + 2n + 4(3 + 2T(n/2)) = 3 + 2n + 12 + 8T(n/2)
= 15 + 2n + 8T(n/2)
= 15 + 2n + 8(3 + 2T(n/2)) = 15 + 2n + 24 + 16T(n/2)
= 39 + 2n + 16T(n/2)
= 39 + 2n + 16(3 + 2T(n/2)) = 39 + 2n + 48 + 32T(n/2)
= 87 + 2n + 32T(n/2)
= 87 + 2n + 32(3 + 2T(n/2)) = 87 + 2n + 96 + 64T(n/2)
= 183 + 2n + 64T(n/2)
= 183 + 2n + 64(3 + 2T(n/2)) = 183 + 2n + 192 + 128T(n/2)
= 375 + 2n + 128T(n/2)
= 375 + 2n + 128(3 + 2T(n/2)) = 375 + 2n + 384 + 256T(n/2)
= 759 + 2n + 256T(n/2)
= 759 + 2n + 256(3 + 2T(n/2)) = 759 + 2n + 768 + 512T(n/2)
= 1527 + 2n + 512T(n/2)
= 1527 + 2n + 512(3 + 2T(n/2)) = 1527 + 2n + 1536 + 1024T(n/2)
= 3063 + 2n + 1024T(n/2)
= 3063 + 2n + 1024(3 + 2T(n/2)) = 3063 + 2n + 3072 + 2048T(n/2)
= 6135 + 2n + 2048T(n/2)
= 6135 + 2n + 2048(3 + 2T(n/2)) = 6135 + 2n + 6144 + 4096T(n/2)
= 12279 + 2n + 4096T(n/2)
= 12279 + 2n + 4096(3 + 2T(n/2)) = 12279 + 2n + 12288 + 8192T(n/2)
= 24567 + 2n + 8192T(n/2)
= 24567 + 2n + 8192(3 + 2T(n/2)) = 24567 + 2n + 24576 + 16384T(n/2)
= 49143 + 2n + 16384T(n/2)
= 49143 + 2n + 16384(3 + 2T(n/2)) = 49143 + 2n + 49152 + 32768T(n/2)
= 98295 + 2n + 32768T(n/2)
= 98295 + 2n + 32768(3 + 2T(n/2)) = 98295 + 2n + 98304 + 65536T(n/2)
= 196599 + 2n + 65536T(n/2)
= 196599 + 2n + 65536(3 + 2T(n/2)) = 196599 + 2n + 196608 + 131072T(n/2)
= 393197 + 2n + 131072T(n/2)
= 393197 + 2n + 131072(3 + 2T(n/2)) = 393197 + 2n + 393216 + 262144T(n/2)
= 786393 + 2n + 262144T(n/2)
= 786393 + 2n + 262144(3 + 2T(n/2)) = 786393 + 2n + 786432 + 524288T(n/2)
= 1572785 + 2n + 524288T(n/2)
= 1572785 + 2n + 524288(3 + 2T(n/2)) = 1572785 + 2n + 1572864 + 1048576T(n/2)
= 3145569 + 2n + 1048576T(n/2)
= 3145569 + 2n + 1048576(3 + 2T(n/2)) = 3145569 + 2n + 3145728 + 2097152T(n/2)
= 6291137 + 2n + 2097152T(n/2)
= 6291137 + 2n + 2097152(3 + 2T(n/2)) = 6291137 + 2n + 6291344 + 4194304T(n/2)
= 12582273 + 2n + 4194304T(n/2)
= 12582273 + 2n + 4194304(3 + 2T(n/2)) = 12582273 + 2n + 12582656 + 8388608T(n/2)
= 25164545 + 2n + 8388608T(n/2)
= 25164545 + 2n + 8388608(3 + 2T(n/2)) = 25164545 + 2n + 25165216 + 16777216T(n/2)
= 50329089 + 2n + 16777216T(n/2)
= 50329089 + 2n + 16777216(3 + 2T(n/2)) = 50329089 + 2n + 50329632 + 33554432T(n/2)
= 100658177 + 2n + 33554432T(n/2)
= 100658177 + 2n + 33554432(3 + 2T(n/2)) = 100658177 + 2n + 100659392 + 67108864T(n/2)
= 201316353 + 2n + 67108864T(n/2)
= 201316353 + 2n + 67108864(3 + 2T(n/2)) = 201316353 + 2n + 201317712 + 134217728T(n/2)
= 402632705 + 2n + 134217728T(n/2)
= 402632705 + 2n + 134217728(3 + 2T(n/2)) = 402632705 + 2n + 402635456 + 268435456T(n/2)
= 805265410 + 2n + 268435456T(n/2)
= 805265410 + 2n + 268435456(3 + 2T(n/2)) = 805265410 + 2n + 805268192 + 536870912T(n/2)
= 1610530820 + 2n + 536870912T(n/2)
= 1610530820 + 2n + 536870912(3 + 2T(n/2)) = 1610530820 + 2n + 1610534048 + 1073741824T(n/2)
= 3221061640 + 2n + 1073741824T(n/2)
= 3221061640 + 2n + 1073741824(3 + 2T(n/2)) = 3221061640 + 2n + 3221065888 + 2147483648T(n/2)
= 6442123280 + 2n + 2147483648T(n/2)
= 6442123280 + 2n + 2147483648(3 + 2T(n/2)) = 6442123280 + 2n + 6442127072 + 4294967296T(n/2)
= 12884246560 + 2n + 4294967296T(n/2)
= 12884246560 + 2n + 4294967296(3 + 2T(n/2)) = 12884246560 + 2n + 12884250880 + 8589934592T(n/2)
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= 25768493120 + 2n + 8589934592(3 + 2T(n/2)) = 25768493120 + 2n + 25768497408 + 17179869184T(n/2)
= 51536986240 + 2n + 17179869184T(n/2)
= 51536986240 + 2n + 17179869184(3 + 2T(n/2)) = 51536986240 + 2n + 51536991680 + 34359738368T(n/2)
= 103073972480 + 2n + 34359738368T(n/2)
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= 206147944960 + 2n + 68719476736T(n/2)
= 206147944960 + 2n + 68719476736(3 + 2T(n/2)) = 206147944960 + 2n + 206147983680 + 137438953472T(n/2)
= 412295889920 + 2n + 137438953472T(n/2)
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= 824591779840 + 2n + 274877906944T(n/2)
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= 3298367119360 + 2n + 1099511627776T(n/2)
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= 211095495639040 + 2n + 70368744177664T(n/2)
= 211095495639040 + 2n + 70368744177664(3 + 2T(n/2)) = 211095495639040 + 2n + 211095498918400 + 140737488355328T(n/2)
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= 422190991278080 + 2n + 140737488355328(3 + 2T(n/2)) = 422190991278080 + 2n + 422190998918400 + 281474976710656T(n/2)
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= 844381982556160 + 2n + 281474976710656(3 + 2T(n/2)) = 844381982556160 + 2n + 844381998918400 + 562949953421312T(n/2)
= 1688763965112320 + 2n + 562949953421312T(n/2)
= 1688763965112320 + 2n + 562949953421312(3 + 2T(n/2)) = 1688763965112320 + 2n + 1688763998918400 + 1125899906842624T(n/2)
= 3377527930224640 + 2n + 1125899906842624T(n/2)
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= 216161787534376960 + 2n + 72057594037927936T(n/2)
= 216161787534376960 + 2n + 72057594037927936(3 + 2T(n/2)) = 216161787534376960 + 2n + 216161791998918400 + 144115188075855872T(n/2)
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= 27668708804400250880 + 2n + 9223372036854775808T(n/2)
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= 113331031262823427604480 + 2n + 37778931862957161709568T(n/2)
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= 226662062525646855208960 + 2n + 75557863725914323419136T(n/2)
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= 453324

ATD: Stacks (cont)

Limitations: max size must be defined prior

pushing to a full stack causes implementation specific error

BinarySearch

```
BinarySearch(A[0..N-1],
value) {
    low = 0
    high = N - 1
    while (low
<= high) {
        mid =
        (low + high) / 2
        if
        (A[mid] > value)
            high = mid - 1
        else
        if (A[mid] < value)
            low = mid + 1
        else
            return mid
    }
    return
not_found // value would
be inserted at index " -
low "
```

Invariants:

value > A[i] for all i < low
value < A[i] for all i > high

Worst case performance: $O(\log n)$

Best case performance: $O(1)$

Average case performance:
 $O(\log n)$

BinarySearch (Recursive)

```
int bsearch(int[] A, int
i, int j, int x) {
    if (i < j) {
        int e = [(i+j)/2];
        if (A[e] > x) {
            return bsearch(A, i, e-
1);
        } else if (A[e] < x) {
            return bsearch(A, e+
1, j);
        } else {
            return e;
        }
    } else { return -1; }
}
```

Time Complexity: $\log(\text{base } 2)(n)$

Insertion Sort (Iterative)

```
For i ← 1 to length(A) -
1
    j ← i
    while j > 0 and A[j-1]
> A[j]
        swap A[j] and A[j-1]
    j ← j - 1
end while
end for
```

Time complexity: $O(n^2)$

Merge-then-sort

Algorithm

ListIntersection (A,m,
B,n)

Input: Same as before

Output: Same as before

inter ← 0

Array C[m+n];

Merge-then-sort (cont)

```
> for i ← 0 to m-1 do C[i] ← A[i];
for i ← 0 to n-1 do C[i+m] ← B[i];
C ← sort( C, m+n );
ptr ← 0
while ( ptr < m+n-1 ) do {
    if ( C[ptr] = C[ptr+1] ) then {
        inter ← inter+1
    }
    ptr ← ptr+2
    }
    else ptr ← ptr+1
}
return inter
```

Time Complexity: $(m+n) * (\lceil \log(-m+n) \rceil) + m + n - 1$

MergeSort (Recursive)

```
MergeSort (A, p, r) //
sort A[p..r] by divide &
conquer
if p < r
    then q ← ⌊ (p+r)/2 ⌋
        MergeSort
        (A, p, q)
        MergeSort
        (A, q+1, r)
        Merge (A, p,
q, r)
```

Time Complexity: $2T(n/2) + k \cdot n + c'$

Primitive Operations

assignment

calling method

returning from method

arithmetic

comparisons of primitive types

conditional

indexing into array

following object reference

Assume each primitive
operation holds the same value
= 1 primitive operation

Prove Big - Oh

Prove big-Oh

$f(n) = n^2 + 2n + 1$ is $O(n^2)$
 $g(n) = n^2$

Show that $\frac{f(n)}{g(n)} \leq c$ for all $n \geq n_0$

$$\frac{n^2 + 2n + 1}{n^2} \leq c$$

$$\frac{n^2 + 2n + 1}{n^2} \leq \frac{n^2 + 2n + 1}{n^2} \quad \forall n \geq 1$$

$$\frac{(1+2/n+1/n^2)}{1} \leq \frac{n^2 + 2n + 1}{n^2}$$

$$4 \geq \frac{n^2 + 2n + 1}{n^2}$$

$$\frac{n^2 + 2n + 1}{n^2} \leq 4 \quad \forall n \geq 1$$

Thus $f(n) = O(g(n))$